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Technical Report No. 29

LINEAR THEORY OF MICROPOLAR ELASTICITY

by

A. Cemal Eringen

to

Office of Naval Research

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Contract Nonr-1100(23)

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ABSTRACT:

Equations of motion, constitutive equations and boundary conditions are presented for a special class of micro-elastic materials called Micropolar Solids. These solids respond to micro-rotational motions and spin inertia and can support couple stress and distributed body couples. The couple stress theory is shown to emanate as a special case of the present theory when the motion is constrained so that micro- and macro-rotations coincide. Several energy and uniqueness theorems are given.

1. INTRODUCTION

The theory of micromorphic materials introduced by Eringen and his co-workers [1] to [5] deals with a class substance which exhibits certain microscopic effects arising from the local structure and micromotions of the media. These materials can support stress moments and body moments and are influenced by the spin inertia. The general theory is however very complicated and even in the case of constitutively linear elastic solids [2] the differential equations are not easily amenable to solution.

To simplify the theory Eringen and Suhubi [2] introduced a theory of couple stress. In view of the fact that in the abundant recent literature the terminology of couple stress is used in a different context we thought it may save the reader from confusion if we name our theory "Micropolar elasticity". In contrast to the couple stress theory in micropolar elasticity all components of the assymmetric stress tensor are determinate and the motion of the media is fully described when the deformation and micro-rotation vectors are known. The concept of micro-rotation and the corresponding field equations are, of course, totally absent in the couple stress theory.

Physically, solids that are composed of dipole macromolecules may be adequately represented by the model of the micropolar elasticity. Fibrous materials and some granular and porous solids may also fall into the domain of this theory.

In Art. 2 we recapitulate the linear theory of microelasticity given in [2] briefly in slightly different form. Art. 3 is devoted to the derivations of the basic equations and boundary conditions of micropolar elasticity. In Art. 4 it is shown that the couple stress theory results as a special case of the micropolar theory when the motion is constrained so that micro- and macro-rotations coincide.

Art. 5 is devoted to a discussion of energy. Here we determine the restrictions that must be imposed on the material constants in order that the internal energy be non-negative. Two theorems on uniqueness, one for the dynamical solutions, one for the static case are given in Art. 6.

2. BASIC EQUATIONS

The theory of micro-elastic solids formulated in [1] and [2] are based on a set of laws of motion and constitutive equations some which are new to the mechanics of continua and others are modifications and extensions of the well-known principles. The basic laws are:

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \qquad \text{in } \mathcal{V} \tag{2.1}$$

Conservation of micro-inertia, [3]:

$$\frac{\partial i_{km}}{\partial t} + i_{km,r} v_r - i_{rm} \nu_{rk} - i_{kr} \nu_{rm} = 0 \qquad \text{in } \mathcal{V} \tag{2.2}$$

Balance of momentum:

$$t_{k\ell,k} + \rho(f_\ell - \dot{v}_\ell) = 0 \qquad \text{in } \mathcal{V} \tag{2.3}$$

Balance of first stress moments:

$$t_{m\ell} - s_{m\ell} + \lambda_{k\ell m,k} + \rho(\ell_{\ell m} - \dot{\sigma}_{\ell m}) = 0 \qquad \text{in } \mathcal{V} \tag{2.4}$$

Conservation of energy:

$$\rho\dot{\epsilon} = t_{k\ell} v_{\ell,k} + (s_{k\ell} - t_{k\ell}) \nu_{k\ell} + \lambda_{k\ell m} \nu_{m\ell,k} + q_{k,k} + \rho h \qquad \text{in } \mathcal{V} \tag{2.5}$$

Principle of entropy:

$$\rho\Gamma \equiv \rho\dot{\eta} - \left(\frac{q_k}{\theta}\right)_{,k} - \frac{\rho h}{\theta} \geq 0 \qquad \text{in } \mathcal{V} \qquad (2.6)$$

This inequality is being axiomatized to be valid for all independent processes. In these equations

ρ = mass density

v_k = velocity vector

i_{km} = i_{mk} = micro-inertia moments

$\nu_{k\ell}$ = gyration tensor

$t_{k\ell}$ = stress tensor

f_ℓ = body force per unit mass

$s_{k\ell}$ = $s_{\ell k}$ = micro-stress average

$\lambda_{k\ell m}$ = the first stress moments

$\ell_{\ell m}$ = the first body moments per unit mass

$\dot{\sigma}_{\ell m}$ = inertial spin

ϵ = internal energy density per unit mass

q_k = heat vector directed outward of the body

h = heat source per unit mass

η = entropy per unit mass

θ = temperature

Throughout the present paper a rectangular coordinate system (x_1, x_2, x_3) is employed, Fig. 1. An index followed by a comma represents partial differentiation with respect to space variables and a superposed dot indicates material differentiation, i.e.

$$v_{k,\ell} \equiv \frac{\partial v_k}{\partial x_\ell} \quad , \quad \dot{v}_k \equiv \frac{\partial v_k}{\partial t} + v_{k,\ell}\, v_\ell \tag{2.7}$$

Here and throughout this paper repeated indices denote summation over the range (1,2,3). The foregoing equations are expressed in Eulerian representation. However, in the linear theory of solids the difference between the Eulerian and material representations disappears since in calculating the time rates the convective terms are ignored. Expressions (2.1) to (2.6) are valid at all parts of the body B having volume $\mathcal{V}$ and surface $\mathcal{S}$, except at finite number of discontinuity surfaces, lines and points. At the surface $\mathcal{S}$ of the body we have the boundary conditions

$$t_{k\ell}\, n_k = t_{(\underset{\sim}{n})\ell} \qquad \text{on } \mathcal{S} \tag{2.8}$$

$$\lambda_{k\ell m}\, n_k = \lambda_{(\underset{\sim}{n})\ell m} \qquad \text{on } \mathcal{S} \tag{2.9}$$

where $\underset{\sim}{n}$ is the exterior normal to $\mathcal{S}$ and $\underset{\sim}{t}_{(\underset{\sim}{n})}$ and $\underset{\sim}{\lambda}_{(\underset{\sim}{n})}$ are respectively the surface tractions and surface moments acting on $\mathcal{S}$. For the spin inertia we have the kinematical relation

$$\dot{\sigma}_{k\ell} = i_{m\ell}(\dot{\nu}_{mk} + \nu_{nk}\, \nu_{mn}) \tag{2.10}$$

We note that while equations (2.1) and (2.3) are well-known from the classical theory, six equations (2.2) are entirely new and nine equations (2.4) and the energy equation (2.5) are modified and extended forms of the corresponding limited axioms. In fact the skew-symmetric part of (2.4) is certainly new and, of course, the kinematical variables $\nu_{k\ell}$ and $i_{k\ell}$ do not enter into the classical continuum mechanics.

If we exclude the phenomena of heat conduction, in the present theory, the determination of motion requires the determination of the following nineteen unknowns

$$\rho(\underset{\sim}{x},t)\ ,\quad i_{k\ell}(\underset{\sim}{x},t)\ ,\quad v_k(\underset{\sim}{x},t)\ ,\quad \nu_{k\ell}(\underset{\sim}{x},t) \tag{2.11}$$

as against the four unknowns ρ and v_k of the classical theory.

The basic equations (2.1) to (2.5) are valid for all types of media independent of their constitutions. The character of the media are reflected through the constitutive equations. For micro-elastic solid in [2] we gave sets of constitutive equations valid for nonlinear deformations. Here produce only the linear theory relevant to the present work. These are

$$\begin{aligned} t_{k\ell} = {} & [(\lambda_1 + \tau)e_{rr} + \eta_1\, \epsilon_{rr}]\, \delta_{k\ell} + 2(\mu_1 + \sigma_1)e_{k\ell} \\ & + \kappa_1\, \epsilon_{\ell k} + \nu_1\, e_{k\ell} \end{aligned} \tag{2.12}$$

$$\begin{aligned} s_{k\ell} = {} & [(\lambda_1 + 2\tau)e_{rr} + (2\eta_1 - \tau)\epsilon_{rr}]\, \delta_{k\ell} + 2(\mu_1 + 2\sigma_1)e_{k\ell} \\ & + (\nu_1 + \kappa_1 - \sigma_1)\, (\epsilon_{k\ell} + \epsilon_{\ell k}) \end{aligned} \tag{2.13}$$

$$\begin{aligned} \lambda_{k\ell m} = {} & \tau_1(\varphi_{kr,r}\, \delta_{m\ell} + \varphi_{rr,\ell}\, \delta_{mk}) + \tau_2(\varphi_{rk,r}\, \delta_{m\ell} \\ & + \varphi_{rr,m}\, \delta_{k\ell}) + \tau_3\, \varphi_{rr,k}\, \delta_{m\ell} + \tau_4\, \varphi_{\ell r,r}\, \delta_{mk} \\ & + \tau_5(\varphi_{r\ell,r}\, \delta_{mk} + \varphi_{mr,r}\, \delta_{k\ell}) + \tau_6\, \varphi_{rm,r}\, \delta_{k\ell} \\ & + \tau_7\, \varphi_{\ell m,k} + \tau_8(\varphi_{mk,\ell} + \varphi_{k\ell,m}) + \tau_9\, \varphi_{\ell k,m} \\ & + \tau_{10}\, \varphi_{m\ell,k} + \tau_{11}\, \varphi_{km,\ell} \end{aligned} \tag{2.14}$$

where λ_1, τ, η_1, μ_1, σ_1, κ_1, ν_1 and τ_1 to τ_{11} are material constants and $e_{k\ell}$ and $\epsilon_{k\ell}$ are respectively the strain tensor and the microstrain tensor of the linear theory defined by

$$e_{k\ell} = \frac{1}{2}(u_{k,\ell} + u_{\ell,k}) \tag{2.15}$$

$$\epsilon_{k\ell} = u_{k,\ell} + \varphi_{k\ell} \tag{2.16}$$

It can be shown that [2] the arc length and the angle changes can be calculated once these quantities and $\varphi_{k\ell,m}$ are known[1]. For the linear theory we also have

$$\nu_{k\ell} = \dot{\varphi}_{k\ell} \tag{2.17}$$

A thermodynamic discussion can be made[2] showing that for a non-heat conducting media

$$\underset{\sim}{q} = 0 \quad , \quad \eta = -\frac{\partial\psi}{\partial\theta} \tag{2.18}$$

where $\psi = \epsilon - \theta\eta$ is the free energy which depends on ρ^{-1} and θ only. Alternatively

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[1] $\nu_{k\ell}$ and $\varphi_{k\ell}$ used here correspond to $\nu_{\ell k}$ and $\varphi_{\ell k}$ of [1] and [2] and $\nu_{k\ell}$ and $\varphi_{k\ell}$ of [3] and [5].

[2] For these and other interesting results we refer the reader to [1], [3] and [5].

$$\underset{\sim}{q} = 0 \quad , \qquad \theta = \frac{\partial \epsilon}{\partial \eta}\bigg|_{\rho} \tag{2.19}$$

where ϵ is considered to be a function of η and ρ^{-1}.

Upon substitution of (2.15) and (2.16) into (2.12) and (2.13) and the result and (2.14) into (2.3) and (2.4) together with (2.1), (2.2), (2.10), (2.17) and $\underset{\sim}{v} = \dot{\underset{\sim}{u}}$ we obtain just the right number of partial differential equations to determine the nineteen unknowns

$$\rho(\underset{\sim}{x},t) \quad , \quad i_{k\ell}(\underset{\sim}{x},t) \quad , \; u_k(\underset{\sim}{x},t) \quad \text{and } \varphi_{k\ell}(\underset{\sim}{x},t).$$

Under appropriate boundary and initial conditions these partial differential equations should be adequate to predict the behavior of micro-elastic solids.

The theory outlined above is too complicated for application. The theory of micropolar elasticity presented below, however, not only brings some elegance into the picture but also is simple enough for mathematical treatments.

3. THEORY OF MICROPOLAR ELASTICITY

Definition. A body will be called micropolar elastic solid if for all motions.

$$\lambda_{k\ell m} = -\lambda_{km\ell} \quad , \quad \varphi_{k\ell} \quad -\varphi_{\ell k} \tag{3.1}$$

The motion of micropolar solids appears to have important implications. Such materials are physically realistic. They exhibit micro-rotational effects and can support body and surface couples. Materials consisting of dumbbell macromolecules belong to this class. Some fibrous and granular media are also approximated by this model.

By use of the skew symmetric properties of $\underset{\sim}{\lambda}$ and $\underset{\sim}{\varphi}$ the basic equations of such continua can be simplified a great deal. We first define couple stress tensor m_{kr}, micro-rotation vector φ_r, micro-angular velocity $\underset{\sim}{\nu}$, micro-rotation inertia $\dot{\sigma}_r$ and body couple ℓ_r by

$$\begin{aligned}
m_{kr} &\equiv -\epsilon_{r\ell m}\lambda_{k\ell m} \\
\varphi_r &\equiv \frac{1}{2}\epsilon_{rk\ell}\varphi_{k\ell} \\
\nu_r &\equiv \frac{1}{2}\epsilon_{rk\ell}\nu_{k\ell} \\
\dot{\sigma}_r &\equiv -\epsilon_{rk\ell}\dot{\sigma}_{k\ell} \\
\ell_r &\equiv -\epsilon_{rk\ell}\ell_{k\ell}
\end{aligned} \tag{3.2}$$

The couple stress tensor m_{kr} has the same sign convention as the stress tensor t_{kr}, Fig. 2.

Upon using (3.2) in (2.4) and (2.5) these equations can be expressed respectively as [5].

$$m_{rk,r} + \epsilon_{k\ell r}\, t_{\ell r} + \rho(\ell_k - \dot{\sigma}_k) = 0 \tag{3.3}$$

$$\begin{aligned} \rho\dot{\epsilon} = {} & t_{k\ell}(v_{\ell,k} - \epsilon_{k\ell r}\, \nu_r) + m_{k\ell}\, \nu_{\ell,k} \\ & + q_{k,k} + \rho h \end{aligned} \tag{3.4}$$

The boundary conditions (2.9) can be written as

$$m_{rk}\, n_r = m_{(n)k} \qquad \text{on} \quad \mathscr{S} \tag{3.5}$$

where $m_{(n)k}$ is the surface couple acting on $\mathscr{S}$.

Similarly the constitutive equations for $\underset{\sim}{t}$ and $\underset{\sim}{\lambda}$ can be reduced to simpler forms. In simplifying the expressions of $\underset{\sim}{\lambda}$ we make use of the skew-symmetry property of $\underset{\sim}{\lambda}$ and $\underset{\sim}{\varphi}$ as expressed by (3.1) and employ the expressions (3.2). This leads to the following restrictions on the coefficients τ_α.

$$\tau_1 = \tau_2 \quad , \quad \tau_4 = \tau_6 \quad , \quad \tau_9 = \tau_{11} \tag{3.6}$$

Using $(3.2)_1$ and $(3.2)_2$ and introducing a set of new elastic coefficients by

$$\begin{aligned} \alpha &\equiv 2(\tau_9 - \tau_8) \quad , \qquad \beta \equiv 2(\tau_4 - \tau_5) \\ \gamma &\equiv 2(-\tau_4 + \tau_5 - \tau_7 + \tau_8 - \tau_9 + \tau_{10}) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \lambda &\equiv \lambda_1 + \tau + \eta_1 \quad , \quad \mu \equiv \mu_1 + \sigma_1 + \nu_1 \\ \kappa &\equiv \kappa_1 - \nu_1 \end{aligned} \tag{3.8}$$

we finally arrive at the constitutive equations

$$t_{k\ell} = \lambda u_{r,r}\ \delta_{k\ell} + \mu(u_{k,\ell} + u_{\ell,k}) + \kappa(u_{\ell,k} - \epsilon_{k\ell r}\varphi_r) \quad (3.9)$$

$$m_{k\ell} = \alpha\varphi_{r,r}\ \delta_{k\ell} + \beta\varphi_{k,\ell} + \gamma\varphi_{\ell,k} \quad (3.10)$$

An alternative form to (3.9) is

$$t_{k\ell} = \lambda e_{rr}\ \delta_{k\ell} + (2\mu + \kappa)\ e_{k\ell} + \kappa\epsilon_{k\ell m}(r_m - \varphi_m) \quad (3.11)$$

where r_k is the rotation vector of the linear theory of elasticity defined by

$$r_k \equiv \frac{1}{2}\epsilon_{k\ell m}\ u_{m,\ell} \quad (3.12)$$

In the present theory the stress average $s_{k\ell}$ disappears from the equations.

For simplicity we consider <u>micro-isotropic solids</u> only. In this case we have

$$i_{k\ell} = i(\underset{\sim}{x},t)\ \delta_{k\ell} \quad (3.13)$$

Carrying this into (2.2) and recalling that $\nu_{k\ell} = -\nu_{\ell k}$ we obtain

$$\frac{Di}{Dt} = 0 \quad \text{or } i = \text{constant} = j/2 \quad \text{on material lines} \quad (3.14)$$

With this then the expression of inertial rotation becomes

$$\dot{\sigma}_k = j\dot{\nu}_k = j\ddot{\varphi}_k \quad (3.15)$$

The field equations of linear micropolar elasticity are obtained by substituting (3.9), (3.10) into (2.3) and (3.3). Hence[1]

$$(\lambda + \mu)u_{\ell,\ell k} + (\mu + \kappa)u_{k,\ell\ell} + \kappa\epsilon_{k\ell m}\varphi_{m,\ell} + \rho(f_k - \ddot{u}_k) = 0 \quad (3.16)$$

$$(\alpha + \beta)\varphi_{\ell,\ell k} + \gamma\varphi_{k,\ell\ell} + \kappa\epsilon_{k\ell m}u_{m,\ell} - 2\kappa\varphi_k + \rho(\ell_k - j\ddot{\varphi}_k) = 0 \quad (3.17)$$

In the linear theory the density ρ is to be treated as constant so that no need arises for the use of the continuity equation (2.1). Generally the integral of this equation for the linear theory is used for the purpose of calculating small changes in density after the displacement field is known. This is given by

$$\rho_0/\rho \cong 1 + u_{k,k} \quad (3.18)$$

The energy equation (3.4) with the use of (3.9) and (3.10) becomes

$$\begin{aligned} \rho\dot{\epsilon} &= \lambda e_{kk}d_{\ell\ell} + (2\mu + \kappa)e_{k\ell}d_{\ell k} + 2\kappa(r_k - \varphi_k)(\omega_k - \nu_k) \\ &+ \alpha\varphi_{k,k}\,\nu_{\ell,\ell} + \beta\varphi_{k,\ell}\nu_{\ell,k} + \gamma\varphi_{\ell,k}\,\nu_{\ell,k} \end{aligned} \quad (3.19)$$

where ω_k is the vorticity vector defined by

$$\omega_k \equiv \frac{1}{2}\epsilon_{k\ell m}\,v_{m,\ell} \quad (3.20)$$

Equation (3.19) can be integrated with respect to time to give

---

[1] These equations can be made to coincide respectively with eqs. (6.13) and (6.14) of [2] by redefining material constants and using $(3.2)_2$.

$$\rho\epsilon - \rho_o\epsilon_o = \frac{1}{2} [\lambda\, e_{kk}\, e_{\ell\ell} + (2\mu + \kappa)\, e_{k\ell}\, e_{\ell k} + 2\kappa(r_k - \varphi_k)(r_k - \varphi_k) + \alpha\varphi_{k,k}\, \varphi_{\ell,\ell} + \beta\varphi_{k,\ell}\, \varphi_{\ell,k} + \gamma\varphi_{\ell,k}\, \varphi_{\ell,k} \tag{3.21}$$

where ρ_o and ϵ_o are the mass density and internal energy in the undeformed body. Without loss in generality we may take $\epsilon_o = 0$. In obtaining (3.21) in the spirit of the linear theory we assumed that $\dot{\underset{\sim}{e}} = \underset{\sim}{d}$. An alternative form to (3.21) is

$$\rho\epsilon = \frac{1}{2} \{t_{k\ell}[e_{k\ell} + \epsilon_{k\ell m}\, (r_m - \varphi_m)] + m_{k\ell}\, \varphi_{\ell,k}\} \tag{3.22}$$

4. COUPLE STRESS THEORY

We now examine a special case of the above theory namely the motion with constraints

$$\varphi_k = r_k \tag{4.1}$$

In this case the stress constitutive equations (3.3) reduce to

$$t_{k\ell} = \lambda e_{rr}\, \delta_{k\ell} + (2\mu + \kappa) e_{k\ell}$$

which are identical to those of the classical theory of elasticity provided we replace $\mu + \frac{\kappa}{2}$ by μ. The field equations (3.16) now become

$$(\lambda + \mu + \frac{\kappa}{2})\, u_{\ell,\ell k} + (\mu + \frac{\kappa}{2}) u_{k,\ell\ell} + \rho(f_k - \ddot{u}_k) = 0$$

which are the celebrated equations of Navier with the replacement of μ for $\mu + \kappa/2$. From equations (3.17) $\underset{\sim}{\mu}$ drops out. Since $\underset{\sim}{\varphi}$ is no longer an independent variable (3.17) fully determines the body couple $\underset{\sim}{\ell}$ admissible for this class of motions. With this viewpoint then the present theory goes into the classical theory of linear elasticity.

There is, however, another way of interpretation. Suppose that (4.2) only determines the symmetric part of the stress tensor i.e., write

$$t_{(k\ell)} = \lambda e_{rr}\ \delta_{k\ell} + (2\mu + \kappa) e_{k\ell} \tag{4.2}$$

where a ( ) enclosing the paranthesis indicates the symmetric part of a tensor and similarly a [ ] will be used to indicate the skew-symmetric part of a tensor, i.e.

$$a_{(k\ell)} = \frac{1}{2}(a_{k\ell} + a_{\ell k}) \qquad , \qquad a_{[k\ell]} = \frac{1}{2}(a_{k\ell} - a_{\ell k})$$

We now solve for the antisymmetric part $t_{[k\ell]}$ of the stress tensor from the original equation (3.3) of the balance of moment of momentum. By multiplying (3.3) by ϵ_{kmn} we first put it into the form

$$t_{[k\ell]} + \frac{1}{2}\epsilon_{rk\ell}\, m_{nr,n} - \rho(\ell_{[k\ell]} - \dot{\sigma}_{[k\ell]}) = 0 \tag{4.3}$$

Upon substituting (4.1) into (3.10) and (3.15) with (3.12) used for r_k we get

$$m_{k\ell} = \frac{\gamma}{2}\epsilon_{\ell mn}\, u_{n,mk} \tag{4.4}$$

$$\dot{\sigma}_{[k\ell]} = j\, \ddot{u}_{[k,\ell]} \tag{4.5}$$

Substituting these into (4.3) we find

$$t_{[k\ell]} = \frac{\gamma}{2}(u_{k,\ell nn} - u_{\ell,knn}) + \rho(\ell_{[k\ell]} - j\, \ddot{u}_{[k,\ell]}) \tag{4.6}$$

Using (4.2) and (4.6) we calculate

$$t_{k\ell,k} = t_{(k\ell),k} + t_{[k\ell],k}$$

and substitute it into the equations of motion (2.3). Hence

$$
\begin{aligned}
(\lambda + \mu + \frac{\kappa}{2} + \frac{\gamma}{4}\nabla^2)\, u_{k,\ell k} &+ (\mu + \frac{\kappa}{2} - \frac{\gamma}{4}\nabla^2) u_{\ell,kk} \\
&+ \frac{\rho}{2}(\ell_{k\ell,k} - \ell_{\ell k,k}) + \rho\, f_\ell = \rho(1 - \frac{j}{2}\nabla^2)\, \ddot{u}_\ell \qquad (4.7) \\
&\qquad + \frac{\rho j}{2}\, \ddot{u}_{k,\ell k}
\end{aligned}
$$

where ∇^2 is the laplacian operator in rectangular coordinates, i.e.

$$
\nabla^2 u_\ell \equiv u_{\ell,kk}
$$

Equations (4.7) are the field equations of a theory known as the couple stress theory. The present theory, therefore, for a constrained motion gives the couple stress theory.

5. NON-NEGATIVE INTERNAL ENERGY

In this section we investigate the conditions that the elastic moduli must be subjected to in order that the internal energy density be non-negative. These conditions have important implications in regard to stability problems, wave propagations and uniqueness theorems.

Theorem 1. (non-negative internal energy). The necessary and sufficient conditions for the internal energy to be non-negative are

$$0 \le 3\lambda + 2\mu + \kappa \quad , \quad 0 \le \mu \quad , \quad 0 \le \kappa$$
$$0 \le 3\alpha + 2\gamma \quad , \quad -\gamma \le \beta \le \gamma \quad , \quad 0 \le \gamma \tag{5.1}$$

The sufficiency of (5.1) is proven by merely observing that when these inequalitics hold each one of the following energies constituting (3.21) are non-negative

$$\rho\epsilon_E \equiv \frac{1}{2}\left[\lambda e_{kk} e_{\ell\ell} + (2\mu + \kappa)\, e_{k\ell} e_{\ell k}\right] \tag{5.2}$$

$$\rho\epsilon_R \equiv \kappa(r_k - \varphi_k)(r_k - \varphi_k) \tag{5.3}$$

$$\rho\epsilon_M \equiv \frac{1}{2}\left[\alpha\varphi_{k,k}\varphi_{\ell,\ell} + \beta\varphi_{k,\ell}\varphi_{\ell,k} + \gamma\varphi_{\ell,k}\varphi_{\ell,k}\right] \tag{5.4}$$

resulting in

$$\rho\epsilon = \rho(\epsilon_E + \epsilon_R + \epsilon_M) \ge 0 \tag{5.5}$$

The fact that $\rho\epsilon_E$ is non-negative under the conditions $(5.1)_1$, $(5.1)_2$ are well-known from the classical theory of elasticity.

In fact these conditions are also sufficient to make $\rho\epsilon_E$ non-negative. It is simple to observe that for $r_k \neq \varphi_k$, $\rho\epsilon_R \geq 0$ if and only if $\kappa \geq 0$. To see the same for $\rho\epsilon_M$ we write this expression in the forms.

$$\rho\epsilon_M = \frac{1}{3}(3\alpha + \beta + \gamma)(\varphi_{1,1} + \varphi_{2,2} + \varphi_{3,3})^2 \pm \beta\,[(\bar{\varphi}_{1,2} \pm \bar{\varphi}_{2,1})^2 + (\bar{\varphi}_{1,3} \pm \bar{\varphi}_{3,1})^2 + (\bar{\varphi}_{2,3} \pm \bar{\varphi}_{3,2})^2 + (\gamma \mp \beta)\,[\bar{\varphi}^2_{1,2} + \bar{\varphi}^2_{2,1} + \bar{\varphi}^2_{1,3} + \bar{\varphi}^2_{3,1} + \bar{\varphi}^2_{2,3} + \bar{\varphi}^2_{3,2}] \tag{5.6}$$

where upper signs go together and lower together and

$$\bar{\varphi}_{k,\ell} \equiv \varphi_{r,r}\,\delta_{k\ell} + \bar{\varphi}_{k,\ell}\ , \qquad \bar{\varphi}_{k,k} = 0 \tag{5.7}$$

From (4.6) we see that in either case

$$3\alpha + \beta + \gamma \geq 0 \qquad \beta \geq 0 \qquad \gamma - \beta \geq 0 \tag{5.8}$$

$$3\alpha + \beta + \gamma \geq 0 \qquad \beta \leq 0 \qquad \gamma + \beta \geq 0 \tag{5.9}$$

we have $\rho\epsilon_M \geq 0$ so that (5.1) are sufficient to make $\rho\epsilon_M \geq 0$.

Conditions (5.1) are also necessary for the non-negativeness of $\rho\epsilon$. To see this we first note that $e_{k\ell}$, $r_k - \varphi_k$ and $\varphi_{k,\ell}$ can be varied independently of each other. Since the above three energies (5.2) to (5.4) are uncoupled with respect to these variables each one of these energies must be non-negative independent of each other. Certainly $\kappa \geq 0$ is also necessary for $\rho\epsilon_R$ to be non-negative provided, of course, $r_k \neq \varphi_k$[1]. From classical elasticity theory we also know that $(6.1)_1$ and $(6.1)_2$ are necessary and sufficient conditions for $\rho\epsilon_E \geq 0$. Hence we would like to

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[1]For $r_k = \varphi_k$ (the couple stress theory), κ need not be restricted by $(5.1)_3$. However, in this case by writing 2μ for $2\mu + \kappa$, κ disappears from all equations.

find out if the last three conditions in (6.1) are necessary for $\rho\epsilon_M$ to be non-negative. To this end write (4.4) as a quadratic form in a nine dimensional space, i.e.

$$\rho\epsilon_M = a_{ij}\, y_i\, y_j \qquad , \qquad a_{ij} = a_{ji}$$

where

$$y_1 \equiv \varphi_{1,1} \quad , \quad y_2 \equiv \varphi_{2,2} \quad , \quad y_3 \equiv \varphi_{3,3}$$

$$y_4 \equiv \varphi_{1,2} \quad , \quad y_5 \equiv \varphi_{2,1} \quad , \quad y_6 \equiv \varphi_{2,3}$$

$$y_7 \equiv \varphi_{3,2} \quad , \quad y_8 \equiv \varphi_{3,1} \quad , \quad y_9 \equiv \varphi_{1,3}$$

$$a_{11} = a_{22} = a_{33} = \alpha + \beta + \gamma \qquad a_{12} = a_{13} = \alpha$$

$$a_{45} = a_{67} = a_{89} = \beta$$

$$a_{44} = a_{55} = a_{66} = a_{77} = a_{88} = a_{99} = \gamma$$

all other $a_{ij} = 0$

The characteristic values a_i of a_{ij} are obtained by solving the equation

$$\det\,(a_{ij} - a\delta_{ij}) = 0$$

The nine roots a_i of this equation are

$$a_1 = a_2 = a_3 = \gamma - \beta \; , \qquad a_4 = a_5 = a_6 = a_7 = a_8 = \gamma + \beta$$

$$a_9 = 3\alpha + \beta + \gamma$$

In order $a_{ij} y_i y_j \geq 0$ to be satisfied to all y_i it is necessary (and sufficient) that

$$\gamma - \beta \geq 0 \quad , \quad \gamma + \beta \geq 0$$

$$3\alpha + \beta + \gamma \geq 0$$

which is the proof of the theorem.

6. THE UNIQUENESS

The uniqueness theorem is a statement that the field equations (3.16), (3.17) or in curvilinear coordinates

$$(\lambda + 2\mu + \kappa) \quad \nabla \nabla \cdot \mathbf{u} - (\mu + \kappa) \quad \nabla \times \nabla \times \mathbf{u} + \kappa \nabla \times \boldsymbol{\varphi} \\ + \rho(\mathbf{f} - \ddot{\mathbf{u}}) = 0 \tag{6.1}$$

$$(\alpha + \beta + \gamma) \nabla \nabla \cdot \boldsymbol{\nu} - \gamma \nabla \times \nabla \times \boldsymbol{\varphi} + \kappa \nabla \times \mathbf{u} - 2\kappa\boldsymbol{\varphi} \\ + \rho(\boldsymbol{\ell} - j\ddot{\boldsymbol{\varphi}}) = 0 \tag{6.2}$$

possess unique solutions under certain boundary and initial conditions for certain values of elastic modulii.

As the initial conditions we consider

$$\begin{aligned} \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_o(\mathbf{x}) \qquad \text{in} \quad \mathcal{V} \\ \dot{\mathbf{u}}(\mathbf{x},0) &= \mathbf{v}_o(\mathbf{x}) \end{aligned} \tag{6.3}$$

$$\begin{aligned} \boldsymbol{\varphi}(\mathbf{x},0) &= \varphi_o(\mathbf{x}) \qquad \text{in} \quad \mathcal{V} \\ \dot{\boldsymbol{\varphi}}(\mathbf{x},0) &= \boldsymbol{\nu}_o(x) \end{aligned} \tag{6.4}$$

where $\mathbf{u}_o$, $\mathbf{v}_o$, $\boldsymbol{\varphi}_o$ and $\boldsymbol{\nu}_o$ are prescribed in $\mathcal{V}$. We leave the boundary conditions presently unspecified. Ultimately we intend to show that variety of conditions are possible. For example $\mathbf{u}$ and $\boldsymbol{\varphi}$ may be specified on the boundary surface $\mathcal{S}$ of the body. It is equally permissible to specify the tractions and couples on $\mathcal{S}$ provided that they are in equilibrium. Many mixtures of both are also possible. In fact, all admissible boundary conditions allowing unique solutions must satisfy

$$\bar{t}_{(n)k}\, \dot{\bar{u}}_k + \bar{m}_{(n)k}\, \dot{\bar{\varphi}}_k = 0 \qquad \text{on } \mathcal{S}, \quad t \geq 0 \tag{6.5}$$

where $\dot{\bar{u}}$, $\dot{\bar{\varphi}}$, $\bar{t}_{(n)}$ and $\bar{m}_{(n)}$ are respectively the differences of u, φ, $t_{(n)}$ and $m_{(n)}$ from their respective values on $\mathcal{S}$.

Theorem 2. (Uniqueness). Let the conditions (5.1), $r_k \neq \varphi_k$ and $j \geq 0$ be satisfied in a bounded, regular domain[1] $\mathcal{V}$ of space with boundary $\mathcal{S}$. Then there exists at most one $u(x,t)$ and one $\varphi(x,t)$ both twice continuously differentiable for x in $\mathcal{V} + \mathcal{S}$ and $0 \leq t \leq \infty$, which satisfy (6.1) and (6.2), the initial conditions (6.3), (6.4) and a set of boundary conditions compatible with (6.5).

To prove this theorem suppose that contrary is valid and two solutions

$$u^{(\alpha)} \quad , \quad \varphi^{(\alpha)} \quad , \quad (\alpha = 1, 2) \tag{6.6}$$

exists satisfying (6.1) to (6.5). Let

$$u \equiv u^{(1)} - u^{(2)} \quad , \quad \varphi \equiv \varphi^{(1)} - \varphi^{(2)} \tag{6.7}$$

Then clearly u and φ satisfy (6.1) and (6.2) with $f \equiv \ell \equiv 0$ and the initial conditions

$$u(x,0) = \dot{u}(x,0) = \varphi(x,0) = \dot{\varphi}(x,0) = 0 \tag{6.8}$$

Let $t_{k\ell}$, $m_{k\ell}$, $t_{(n)k}$ and $m_{(n)k}$ be respectively the stress, couple stress, surface traction and surface couple corresponding to the solution (6.7), i.e.

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[1] The terminology "regular domain of space" is used to denote domains for which the Green-Gauss theorem is valid.

$$t_{k\ell} = t^{(1)}_{k\ell} - t^{(2)}_{k\ell} \quad , \quad m_{k\ell} \equiv m^{(1)}_{k\ell} - m^{(2)}_{k\ell} \quad , \quad \ldots \tag{6.9}$$

We compute the rate of the total internal energy

$$\dot{\mathcal{E}} = \frac{d}{dt}\int_{\mathcal{V}} \rho\epsilon \, dv = \int_{\mathcal{V}} \rho\dot{\epsilon} \, dv$$

where we employed the continuity equation $\dot{\overline{\rho \, dv}} = 0$. Using (3.4) with $\underset{\sim}{q} = 0$, $h = 0$, this may be put into the form

$$\dot{\mathcal{E}} = \int_{\mathcal{V}} [(t_{k\ell}v_\ell)_{,k} - t_{k\ell,k}v_\ell - \epsilon_{k\ell r}t_{k\ell}\nu_r$$
$$+(m_{k\ell}\nu_\ell)_{,k} - m_{k\ell,k}\nu_\ell]dv$$

Applying Green-Gauss theorem to the first and fourth term and replacing $t_{k\ell,k}$ and $m_{k\ell,k}$ by their respective values (eqs. 2.3 and (3.3) with $\underset{\sim}{f} = \underset{\sim}{\ell} = 0$) i.e.

$$t_{k\ell,k} = \rho\dot{v}_\ell$$

$$m_{k\ell,k} = -\epsilon_{\ell kr}t_{kr} + \rho j\dot{\nu}_\ell$$

we obtain

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \oint_{\mathcal{S}} (t_{k\ell}v_\ell + m_{k\ell}\nu_\ell)n_k \, da \tag{6.10}$$

where $\underset{\sim}{n}$ is the exterior normal to $\mathcal{S}$ and

$$\mathcal{K} \equiv \frac{1}{2}\int_{\mathcal{V}} \rho(v_kv_k + j\nu_k\nu_k)dv \tag{6.11}$$

is the total kinetic energy. If the boundary conditions are such as to make the right hand side of (6.10) vanish we get

$$\mathcal{E}(t) + \mathcal{K}(t) = C$$

where C is a constant. Since both $\mathcal{E}$ and $\mathcal{K}$ are non-negative we have

$$\mathcal{E}(0) + \mathcal{K}(0) = 0$$

Consequently

$$\mathcal{E}(t) + \mathcal{K}(t) = 0 \quad , \quad 0 \leq t < \infty \tag{6.12}$$

With $j \geq 0$ we have $\mathcal{K}$ non-negative. By virtue of (4.1) $\mathcal{E}$ is non-negative. Therefore we must have

$$\mathcal{E}(t) = 0 \quad , \quad \mathcal{K}(t) = 0 \qquad 0 \leq t < \infty$$

From the second of these it follows that since $\underline{u}(\underline{x},0) = \underline{\varphi}(\underline{x},0) = 0$ it follows that

$$u(\underline{x},t) = 0 \quad , \quad \underline{\varphi}(\underline{x},t) = 0 \quad \text{in } \mathcal{V} \, , \quad 0 \leq t < \infty$$

This means that $\underline{u}^{(1)} = \underline{u}^{(2)}$ and $\underline{\varphi}^{(1)} = \underline{\varphi}^{(2)}$. Hence the proof of the theorem.

Next we investigate the static case. For this we give the following theorem which is the counterpart of the Clapeyron's theorem in the classical elasticity theory.

Theorem 3. If a body is in equilibrium under a given system of body loads $\underline{f}$, $\underline{\ell}$ and surface loads $\underline{t}_{(\underline{n})}$ and $\underline{m}_{(\underline{n})}$, then the internal energy of deformation is equal to one-half of the work of external loads that produce displacements and rotations $\underline{u}$ and $\underline{\varphi}$ from the unstressed state. Mathematically

$$\int_{\mathcal{V}} \rho(f_k u_k + \ell_k \varphi_k)\, dv + \oint_{\mathcal{S}} (t_{(n)k} u_k + m_{(n)k} \varphi_k)\, da = 2\int_{\mathcal{V}} \rho\epsilon\, dv \tag{6.13}$$

The proof of this theorem follows by forming the first integral on the left and using Green-Gauss theorem. For

$$V \equiv -\int_{\mathcal{V}} \rho(f_k u_k + \ell_k \varphi_k)\, dv = \int_{\mathcal{V}} [t_{\ell k,\ell} u_k + (m_{\ell k,\ell} + \epsilon_{k\ell r} t_{\ell r})\varphi_k]\, dv = \int_{\mathcal{V}} (t_{\ell k} u_k + m_{\ell k}\varphi_k)_{,\ell}\, dv - \int_{\mathcal{V}} (t_{k\ell} u_{k,\ell} + m_{\ell k}\varphi_{k,\ell} + \epsilon_{k\ell r} t_{\ell r}\varphi_k) dv$$

By use of Green-Gauss theorem we convert the first integral on the right to a surface integral. Upon using (3.22) in the second we get

$$V = \oint_{\mathcal{S}} (t_{(n)k} u_k + m_{(n)k}\varphi_k)\, da - \int_{\mathcal{V}} 2\rho\epsilon\, dv$$

which is the proof of the theorem.

The above theorem can be used to prove:

Theorem 4. _(Uniqueness of static solutions). Let the conditions (5.1) with $\dot{u} \equiv \dot{\varphi} \equiv 0$ and $\varphi_k \not\equiv r_k$ be satisfied in a bounded, regular domain $\mathcal{V}$ of space with boundary $\mathcal{S}$. Then there exists at most one stress field $t_{k\ell}$ and one couple stress field $m_{k\ell}$ arising from twice continuously differentiable displacement field $\underset{\sim}{u}(\underset{\sim}{x})$ and micro-rotation field $\underset{\sim}{\varphi}(\underset{\sim}{x})$ satisfying_

$$t_{(\underset{\sim}{n})k} = t_{ok} \quad , \quad m_{(\underset{\sim}{n})k} = m_{ok} \qquad \text{on} \quad \mathcal{S}_L \tag{6.14}$$

$$u_k = u_{ok} \quad , \quad \varphi_k = \varphi_{ok} \qquad \text{on} \quad \mathcal{S} - \mathcal{S}_L \tag{6.15}$$

Proof: Let $\underset{\sim}{u}^{(\alpha)}(\underset{\sim}{x})$ and $\underset{\sim}{\varphi}^{(\alpha)}(\underset{\sim}{x})$, $(\alpha = 1,2)$ be two distinct solutions of (6.1) and (6.2)(with $\dot{\underset{\sim}{u}} \equiv \dot{\underset{\sim}{\varphi}} \equiv 0$) satisfying (6.14) and (6.15). In view of the linearity of (6.1) and (6.2)

$$\underset{\sim}{u} \equiv \underset{\sim}{u}^{(1)} - \underset{\sim}{u}^{(2)} \qquad , \qquad \underset{\sim}{\varphi} \equiv \underset{\sim}{\varphi}^{(1)} - \underset{\sim}{\varphi}^{(2)}$$

are also the solution of these equations with vanishing $\underset{\sim}{f}$ and $\underset{\sim}{\ell}$. Consequently by the theorem 3 above we have

$$\oint_{\mathcal{S}} (t_{(\underset{\sim}{n})k} u_k + m_{(\underset{\sim}{n})k}\, \varphi_k)\, da = 2\int_{\mathcal{V}} \rho\epsilon\, dv \tag{6.16}$$

But on $\mathcal{S}_L$ we have $\underset{\sim}{t}_{(\underset{\sim}{n})} = \underset{\sim}{m}_{(\underset{\sim}{n})} = 0$ since both $\underset{\sim}{t}^{(1)}_{(\underset{\sim}{n})}$ and $\underset{\sim}{t}^{(2)}_{(\underset{\sim}{n})}$ have the common value $\underset{\sim}{t}_o$ and $m^{(1)}_{(\underset{\sim}{n})}$ and $m^{(2)}_{(\underset{\sim}{n})}$ the same value $\underset{\sim}{m}_o$. Similarly on $\mathcal{S} - \mathcal{S}_L$, $\underset{\sim}{u} = \underset{\sim}{\varphi} = 0$. Thus (6.16) reduces to

$$\int_{\mathcal{V}} \rho\epsilon\, dv = 0$$

But because of the conditions (5.1) this integral can vanish only if

$$e_{k\ell} = 0 \quad , \quad \varphi_k - r_k = 0 \tag{6.17}$$

According to constitutive equations (3.10) and (3.11) then $t_{k\ell} = m_{k\ell} = 0$. Hence the proof of the theorem.

The integral of $(6.17)_1$ is

$$u_k = \Omega_{k\ell}\, x_\ell + b_k \tag{6.18}$$

where $\Omega_{k\ell} = - \Omega_{\ell k}$ and b_k are constants. Substituting (6.18) into $(6.17)_2$ to calculate r_k and using (3.12) we get

$$\varphi_k = \Omega_k$$

where

$$\Omega_k = \frac{1}{2} \epsilon_{k\ell m} \Omega_{m\ell}$$

is a rigid rotation. We therefore see that the displacement and micro-rotation fields are determined within a rigid rotation and translation. When $\mathscr{S} \neq \mathscr{S}_L \neq 0$ we have the mixed boundary conditions. Since in this case $\underset{\sim}{u} \equiv \underset{\sim}{\varphi} = 0$ on $\mathscr{S} - \mathscr{S}_L$ we see that $\underset{\sim}{\Omega} = \underset{\sim}{b} = 0$ and the fields $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ are uniquely determined.

In the case $\mathscr{S} \equiv \mathscr{S}_L$ we have the traction boundary problem. For this case the fields $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ are determined within a rigid rotation and translation. When $\underset{\sim}{t}_o$ and $\underset{\sim}{m}_o$ are in equilibrium, i.e.

$$\oint_{\mathscr{S}} \underset{\sim}{t}_o \cdot d\underset{\sim}{a} = 0 \ , \qquad \oint_{\mathscr{S}} (\underset{\sim}{x} \times \underset{\sim}{t}_o + \underset{\sim}{m}_o) \cdot d\underset{\sim}{a} = 0 \tag{6.19}$$

then $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ are again uniquely determined. A "Mixed-Mixed" type of boundary condition exists for which the solution of (6.1) and (6.2) is unique. All such class of solutions must satisfy

$$\bar{t}_{(n)k} \bar{u}_k + \bar{m}_{(n)k} \bar{\varphi}_k = 0$$

where $\bar{\underset{\sim}{t}}_{(n)}$, $\bar{\underset{\sim}{m}}_{(n)}$, $\bar{\underset{\sim}{u}}$ and $\bar{\underset{\sim}{\varphi}}$ are the difference of $\underset{\sim}{t}_{(n)}$, $\underset{\sim}{m}_{(n)}$, $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ from their respective values on $\mathscr{S}$.

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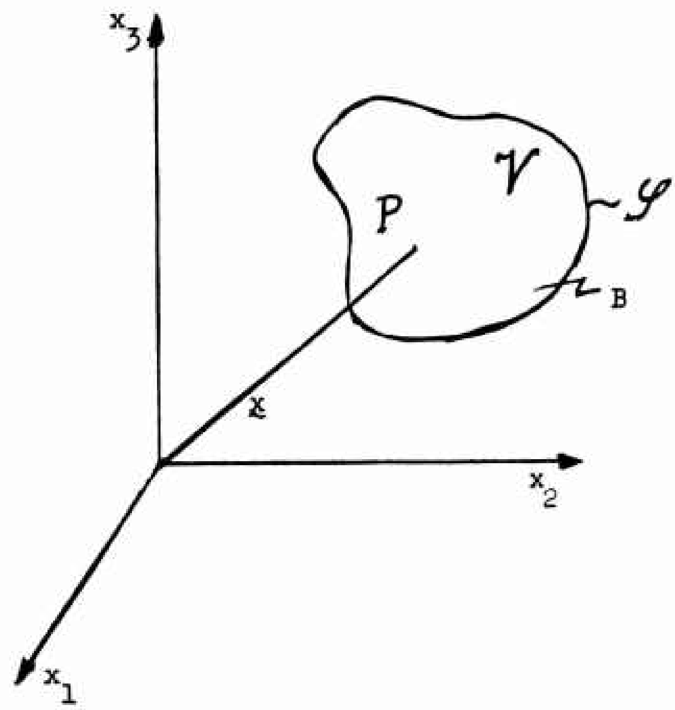

Fig. 1 Coordinates

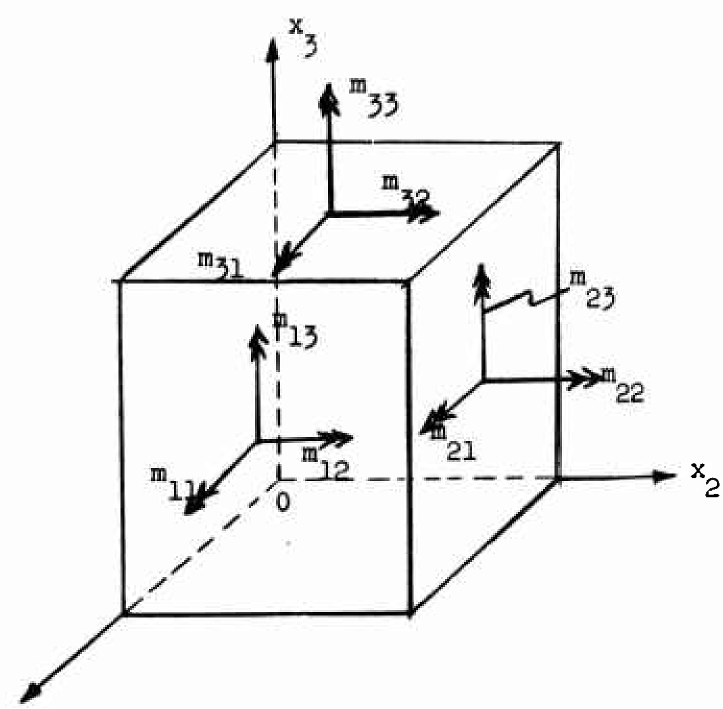

Fig. 2 Positive Couple Stress Components